# Lecture 3: Some simple systems

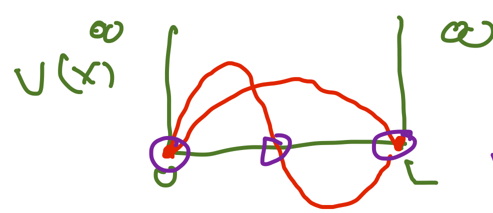
## 5 The free particle

*Read 3-2 to ~~3-5~~* 3-6. In some ways the simplest system is a particle where the potential energy is zero everywhere. Then (in one dimension), setting $k^2 = 2mE/\hbar^2$

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0 \quad \Rightarrow \quad \psi(x) = Ae^{ikx} + Be^{-ikx} \quad \textit{(3-27),(3-33)}$$

Normalization is tricky here (see your text). The two solutions above correspond to a particle moving in either the $+x$ or $-x$ directions.

$\langle p \rangle = \langle e^{-ikx} | -i\hbar \, \partial/\partial x | e^{-ikx} \rangle \sim -\hbar k$




nodes are when $\psi = 0$

## 6 The particle in a box

Suppose we have a system where $V(x) = 0$ inside a box from $x = 0$ to $x = L$, and infinite elsewhere.

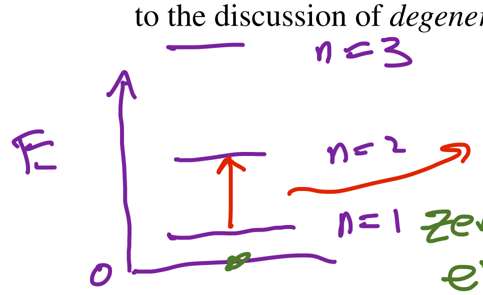
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \qquad k^2 = \frac{2mE}{\hbar^2} \Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \textit{(3-27)}$$

This leads to $\psi(x) = C\sin(kx)$ and to satisfy the boundary conditions, $kL = n\pi$, $n = 1, 2, 3, \ldots$ *(3-37)*. The energy of state $n$ is

$$E_n = n^2\frac{h^2}{8mL^2} \quad \text{or, in two dimensions,} \quad E_n = (n^2 + m^2)\frac{h^2}{8mL^2} \quad \textit{(3-42),(3-46)}$$

Study Figs 3-6 to 3-8 to see results for a particle in a two-dimensional box. Pay careful attention to the discussion of *degeneracy*.

3-quantum numbers in 3D

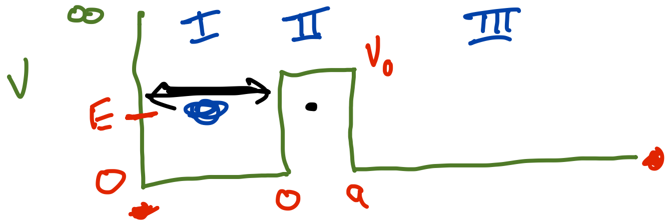



$\Delta E_{2-1} = h\nu$

zero-point energy

$\int_0^L C^2 \sin^2(kx)\,dx = 1$

$\Rightarrow$ determine C by normalization




# 7 Tunneling

See Fig. 3-11.

$$\text{Regions I and III: } \frac{d^2\psi}{dx^2}+\kappa^2\psi = 0 \quad with \quad \kappa^2 = 2mE/\hbar^2$$

$$\text{Region II: } \frac{d^2\psi}{dx^2}-k^2\psi = 0 \quad with \quad k^2 = 2m(V_0-E)/\hbar^2$$

$$\psi_I = Ae^{i\kappa x}+Be^{-i\kappa x}; \quad \psi_{II} = Ce^{kx}+De^{-kx}; \quad \psi_{III} = Fe^{i\kappa x}+\cancel{G}e^{-i\kappa x}$$

no source of particles from x = +∞

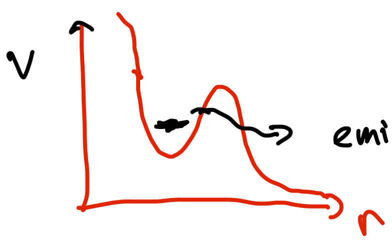
Boundary conditions are:

$\psi_I(0)=\psi_{II}(0)$; $\psi_{II}(a)=\psi_{III}(a)$; plus same for derivatives *(3-55)*

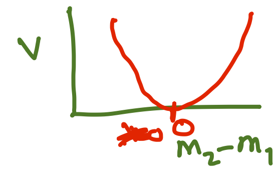
Transmission coefficient is then $\chi = |F|^2/|A|^2$ *(3-56)*. This leads to four simultaneous equations, (3-57), which one can solve for $A$ in terms of $F$ *(3-61)* yielding a formula for $\chi$ in Eq. *(3-62)*. Its essential exponential behavior is this:

$$\chi \sim e^{-2ka} = \exp\left\{-\frac{2a}{\hbar}[2m(V_0-E)]^{1/2}\right\} \quad (3\text{-}63)$$

correspondence principle




emission of α-particles


# 8 The harmonic oscillator

See Fig. 3-16. This is a tough, 9-page, section, and the key is to not get lost in all the details of math. Let's consider the key equations:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}k(x_2 - x_1 - x_{eq})^2$$

## 8.1 separation of center-of-mass motion

• $\hat{H}\Psi(x_1, x_2) = E\Psi(x_1, x_2)$

The Schrodinger equation is (3-75); change variables in (3-76) to get two new equations:

$\begin{cases} x_{com} \\ x = x_2 - x_1 - x_{eq} \end{cases}$

$\Psi(x_1, x_2) = \phi(x_{cm})\psi(x)$

$$-\frac{\hbar^2}{2M}\frac{d^2\phi}{dx^2} = E_{trans}\phi \quad \left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{kx^2}{2}\right)\psi = E_{vib}\psi \quad \text{(3-79),(3-80)}$$

$V \neq 0$ now!

## 8.2 introduce some dimensionless variables

$\bar{H} \sim \hat{H}$ but is dimensionless

$\hat{P}, \hat{\xi}$

aside – $\psi(x) \sim e^{-\alpha x^2}$ is a solution to 3-80

$$\bar{H} = \frac{1}{2}\left(P^2 + \xi^2\right) \quad \bar{H}\psi(\xi) = \tfrac{1}{2}\epsilon\psi(\xi) \quad \text{(3-83),(3-84)}$$

$\hat{A}\hat{B} \neq \hat{B}\hat{A}$

## 8.3 factorize this into "ladder operators"

what are all the other solution?

→ make sure you understand this!

First, note that $A^2 + B^2 = (A + iB)(A - iB) + i[A, B]$ (leaving off the caret symbols). Then define what are called *ladder operators:*

$$F = \frac{\xi + iP}{\sqrt{2}}; \quad F^\dagger = \frac{\xi - iP}{\sqrt{2}}; \quad \Rightarrow \quad \bar{H} = FF^\dagger - \tfrac{1}{2} = F^\dagger F + \tfrac{1}{2} \quad \text{(3-87),(3-89)}$$

Since $\bar{H}$ and $FF^\dagger$ or $F^\dagger F$ only differ by constants, they have the same eigenfunctions:

$$\bar{H}\psi = \tfrac{1}{2}\epsilon\psi \quad (FF^\dagger)\psi = \tfrac{1}{2}(\epsilon + 1)\psi \quad (F^\dagger F)\psi = \tfrac{1}{2}(\epsilon - 1)\psi \quad \text{(3-90)}$$

reason they are called ladder operators

## 8.4 obtain the eigenvalues of the Hamiltonian

Key argument: the energy must be positive, there must be a limit to how far the reduction can go, hence there must be some eigenfuction $\psi_0$ for which $F\psi_0 = 0$. A bit of (complicated) algebra allows us to write the allowed eigenvalues of the original Hamiltonian as:

$$\rightarrow \quad E_v = (v + \tfrac{1}{2})\hbar\omega_0 \quad \text{where} \quad \omega_0 = 2\pi\nu_0 = \left(\frac{k}{\mu}\right)^{1/2} \quad \text{(3-98),(3-99)}$$

## 8.5 obtain the eigenfunctions

We're going to skip this step, but the gory details are in Eqs. *(3-100)* to *(3-121)*. But be sure to study carefully Fig. 3-17.