Singular Value Decomposition (SVD)

$$\begin{array}{c} \\ M \end{array}\!\!\begin{array}{c} N \\ \begin{pmatrix} \\ A \\ \\ \end{pmatrix} \end{array} = \begin{array}{c} N \\ {}^{M}\!\begin{pmatrix} \\ U \\ \\ \end{pmatrix} \end{array} {}_{N}\!\begin{array}{c} N \\ \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_N \end{pmatrix} \end{array}{}^{N}\! \begin{array}{c} N \\ \begin{pmatrix} V^T \end{pmatrix} \end{array} \qquad M > N \tag{1}$$

$$M\begin{array}{c} N \\ \begin{pmatrix} \quad A \quad \end{pmatrix} \end{array} = \begin{pmatrix} \quad U \quad \end{pmatrix} \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_N \end{pmatrix} \begin{pmatrix} V^T \end{pmatrix} \qquad M < N$$

(for $M \geq N$)

$$\underset{\approx}{V}^T \underset{\approx}{V} = \underset{\approx}{V}\,\underset{\approx}{V}^T = \underset{\approx}{I}_{N\times N} \; ; \qquad \underset{\approx}{U}^T \underset{\approx}{U} = \underset{\approx}{I}_{N\times N} \; ; \qquad w_i \geq 0 \tag{2}$$

define a pseudo-inverse $\quad \underset{\approx}{A}^{-1} = \underset{\approx}{V}\,(\text{diag}\; {}^{1}/_{w_j})\,\underset{\approx}{U}^T \quad$ (3)

$$\underset{\approx}{A}^{-1}\underset{\approx}{A} = \underset{\approx}{V}\,(\text{diag}\; {}^{1}/_{w_j})\,\underset{\approx}{U}^T\underset{\approx}{U}\,(\text{diag}\; w_j)\,\underset{\approx}{V}^T = \underset{\approx}{I}_{N\times N} \tag{4}$$

to solve a set of linear equations: $(M>N)$

$$\begin{pmatrix} \\ A \\ \\ \end{pmatrix}\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} \\ b \\ \\ \end{pmatrix} \Rightarrow \underset{\sim}{x} = \underset{\approx}{A}^{-1}\underset{\sim}{b} = \underset{\approx}{V}\,(\text{diag}\; {}^{1}/_{w_j})\,\underset{\approx}{U}^T\underset{\sim}{b} \tag{5}$$

(a) Suppose one of the w_j's is zero : substitute ${}^{1}/_{w_j} \to 0$ (!)

(b) " almost zero: "

(c) you want a "smoother" solution: "

(d) From Eq (1) : $A_{ij} = \sum_k w_k U_{ik} V_{jk}$: to approximate $\underset{\approx}{A}$, set small $w_k \to 0$